

# CONTACT BLOW-UP

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ABSTRACT. We provide various definitions for the contact blow-up. Such different approaches to the contact blow-up are related. Some uniqueness and non-uniqueness results are also provided.

## 1. INTRODUCTION

In his book *Partial Differential Relations* [Gr], M. Gromov proposed<sup>1</sup> a definition of the blow-up surgery operation in the contact category. This article discusses it along with related constructions. Let us summarize his statement. Assume that  $(M, \xi)$  is a cooriented contact manifold and fix a compatible contact form  $\alpha$ , so that  $\ker \alpha = \xi$ . A submanifold  $S \xrightarrow{e} M$  is a contact submanifold if the induced distribution  $\xi_S = e^*(\xi)$  is a contact distribution on  $S$ . This implies that the bundle  $(\xi_S, d(e^*\alpha))$  is a symplectic sub-bundle of  $(\xi|_S, (d\alpha)|_S)$ . The symplectic orthogonal of  $\xi_S$  inside  $\xi|_S$  will be denoted  $\nu_S$  and it is a geometric representative of the normal bundle of  $S$ . Let  $\Psi$  be a diffeomorphic identification of a small tubular neighbourhood  $U_S$  of  $S$  in  $M$  with a small neighbourhood of the zero section  $s_0$  in  $\nu_S$ . Fix a compatible complex structure  $J$  for the symplectic bundle  $(\nu_S, (d\alpha)|_S)$ , which becomes a complex bundle. We are now able to parametrically blow-up all the fibers of the complex bundle. In precise terms, consider the map

$$\phi : \nu_S \setminus \{s_0\} \longrightarrow \nu_S \times \mathbb{P}(\nu_S), \quad v \longmapsto (v, [v]),$$

where  $[v]$  denotes the  $J$ -complex line generated by  $v$ . The map  $\phi$  is a diffeomorphism onto its image. We define the blown-up bundle as the compactified manifold

$$\widetilde{\nu}_S = \phi(\nu_S \setminus \{s_0\}) \cup (\{s_0\} \times \mathbb{P}(\nu_S)).$$

The submanifold  $E_S = \{s_0\} \times \mathbb{P}(\nu_S)$  is called the exceptional divisor. By means of the identification  $\Psi$ , we replace the tubular neighbourhood of  $U_S$  by a tubular neighbourhood of the exceptional divisor, producing a manifold  $\widetilde{M}$  diffeomorphic to  $M \setminus S$  away from  $E_S$ . Note that so far no contact structure has been introduced in  $\widetilde{M}$ .

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<sup>1</sup>See Exercise (c) in page 343.

The previous construction only uses the fact that the normal bundle is a complex bundle and it applies to any category where this is the case. Thus, if the manifolds  $S$  and  $M$  are complex, then there is a blow-up surgery and the blown-up manifold inherits a natural complex structure. If the manifolds  $S$  and  $M$  are symplectic the blow-up manifold can also be endowed with a symplectic structure<sup>2</sup>, see [MS]. In this paper we address the question in the contact case.

M. Gromov conjectured that a contact structure exists in the blown-up manifold if a pair of further hypotheses are satisfied. These are:

- (1) The contact submanifold  $(S, \alpha_S = e^*(\alpha))$  is a Boothby–Wang manifold, see Definition 2.1. Geometrically, this means that the Reeb vector field associated to  $\alpha_S$  has all its orbits periodic with the same period; then the quotient space  $W$  of orbits is a symplectic manifold. The projection map will be denoted by  $\pi : S \rightarrow W$ .
- (2) The normal symplectic bundle  $\nu_S$  is isomorphic to the pull-back through  $\pi$  of a symplectic bundle  $V \rightarrow W$ :  $\nu_S \cong \pi^*V$  as symplectic bundles.

In Section 4 we discuss Gromov’s approach in detail. It turns out that the hypotheses are absolutely precise, however the trivially analogous result is not: the natural construction in complex or symplectic geometry leads to an *inappropriate* situation in contact geometry. Let us explain this in a simple example. Suppose that  $(M, \xi)$  is a 5-dimensional contact manifold and  $S$  is a 1-dimensional contact submanifold, that is a transverse embedded loop. We are in the above hypothesis since the contact manifold  $S$  projects to a point

$$\pi : S \rightarrow \{pt\},$$

and its normal bundle is always trivial, so it satisfies  $\nu_S = \pi^*\mathbb{C}^2$ . If we defined the contact blow-up as a parametric operation using the already existing symplectic blow-up, the exceptional divisor should be  $E = \mathbb{S}^1 \times \mathbb{CP}^1$ . It would be reasonable though to require that, since  $S$  is transverse to the distribution, the fibers of the projection  $E \rightarrow \mathbb{CP}^1$  are also transverse to the contact structure in the blown-up manifold. In this setting, this is not possible: there is no contact distribution on  $\mathbb{S}^1 \times \mathbb{CP}^1$  whose fibers are all transverse, see [Gi]. The inaccuracy arises because it is assumed that the needed surgery is a complex topological blow-up. We will show from 3 different points of view that the right surgery substitutes  $S \cong \mathbb{S}^1$  by the standard contact sphere  $\mathbb{S}^3$ . In such a case, the natural projection map  $\pi : \mathbb{S}^3 \rightarrow \mathbb{CP}^1$  is the Hopf fibration, whose fibers are transverse to the contact structure. In terms of the dynamics, the contact blow-up replaces each orbit of the Reeb vector field in  $S$  by a family of orbits separating complex directions.

The content of the paper is organized as follows. In Section 2, we introduce the classical Boothby–Wang construction [BW]. It will be described in some concrete examples that shall be used later on. Then, three notions of contact blow-up are introduced:

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<sup>2</sup>Far from canonical, there are uniqueness issues.

- (1) The contact blow-up for embedded transverse loops, produced as a surgery operation. This is already introduced in [CPP] and it will be explained in Section 3.
- (2) The contact blow-up *à la Gromov* is the content of Section 4.
- (3) The contact blow-up realized as a contact quotient will be described in Section 5.

These three constructions are inspired by the three alternative constructions for the symplectic blow-up: the *ad hoc* construction with explicit gluings, the description using frame bundles, found in pages 239 and 243 in [MS] respectively, and the symplectic cut procedure discussed in [Le2].

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## 2. BOOTHBY-WANG CONSTRUCTIONS

In this section we recall the Boothby-Wang technique to produce a contact manifold from an integral symplectic manifold as developed in [BW]. We prove a couple of results describing the structure of relevant objects later appearing in the constructions. These will be used to understand the contact topology of the resulting blown-up contact manifold.

**Definition 2.1.** *Let  $(W, \omega)$  be an integral symplectic manifold. The Boothby-Wang manifold  $S_k(W)$  is the contact manifold whose total space is the circle bundle associated to the line bundle induced by  $k\omega \in H^2(W, \mathbb{Z}) \cong [W, BU(1)]$  and its contact structure is defined as the restriction of any connection with curvature  $-ik\omega$  to the circle bundle.*

For the case  $k = 1$  we will sometimes omit the subindex. Note that the topology of the total space varies with the parameter  $k$ . The exact relation is the content of the following

**Lemma 2.2.** *Let  $(W, \omega)$  be a symplectic manifold. Then the Boothby-Wang manifold  $S_1(W)$  is a  $k$ -covering of  $S_k(W)$ .*

*Proof.* This is based on the unitary non-linear map between line bundles

$$L \longrightarrow L^{\otimes k}, \quad u \longmapsto u^{\otimes k}$$

There exists a unitary action of  $\mathbb{Z}_k$  in  $L$  given by

$$\mathbb{Z}/k\mathbb{Z} \times L \longrightarrow L, \quad (c; u) \longmapsto e^{2\pi ic/k} u$$

This action induces the trivial action in  $L^{\otimes k}$  and thus becomes the deck transformation group of a covering between the total spaces of the sphere bundles.  $\square$

In particular  $S_k(\mathbb{CP}^{n-1}) \cong L(k, 1)$  as contact manifolds with the natural contact structures. The total spaces of circle bundles provide interesting examples of contact manifolds, for instance the base  $(T^2, k \cdot dA)$  gives rise to quotients of the Heisenberg group by discrete subgroups  $\Gamma_k$  and thus several examples of contact nilmanifolds different from the 3-torus.

The construction of the blow-up will involve the quotient of the product of two Boothby-Wang manifolds; more generally, we shall describe the behaviour of the Boothby-Wang construction under the Cartesian product. We show that the Boothby-Wang construction and

the Cartesian product *commute*. In precise terms, let  $\mathbb{S}_{(b,a)}(W_1 \times W_2)$  be the Boothby–Wang manifold associated to

$$(W_1 \times W_2, b\pi_1^*\omega_1 + a\pi_2^*\omega_2),$$

the following result characterizes this space:

**Theorem 2.3.** *Let  $(W_1, \omega_1)$  and  $(W_2, \omega_2)$  be symplectic manifolds and  $a, b \in \mathbb{Z}$  a pair of coprime integers. Consider the product  $\mathbb{S}(W_1) \times \mathbb{S}(W_2)$  of the Boothby–Wang manifolds and the action*

$$\varphi_{(a,-b)} : \mathbb{S}^1 \times \mathbb{S}(W_1) \times \mathbb{S}(W_2) \longrightarrow \mathbb{S}(W_1) \times \mathbb{S}(W_2)$$

$$(p, q) \longmapsto \theta \cdot (p, q) = (a\theta \cdot p, -b\theta \cdot q)$$

*Then the space of orbits is a manifold diffeomorphic to  $\mathbb{S}_{(b,a)}(W_1 \times W_2)$ . Further, this manifold carries a contact structure induced by a connection with curvature*

$$b\pi_1^*\omega_1 + a\pi_2^*\omega_2.$$

*Proof.* The action is free and thus the quotient is indeed a manifold. Let  $L(\omega_1)$  and  $L(\omega_2)$  be the complex line bundles associated to the symplectic forms

$$\omega_1 \in H^2(W_1, \mathbb{Z}) \text{ and } \omega_2 \in H^2(W_2, \mathbb{Z}).$$

Consider a pair of trivializing charts  $\{U_j\}$  and  $\{V_k\}$  for the bundles. The charts

$$\{\pi_1^{-1}(U_j) \cap \pi_2^{-1}(V_k)\}_{jk}$$

allow us to trivialize the bundle  $L(b\omega_1 + a\omega_2)$  over  $W_1 \times W_2$  and provide coordinates for the manifold  $L(\omega_1) \times L(\omega_2)$ . Locally consider the map

$$\phi_{jk} : L(\omega_1) \times L(\omega_2) \longrightarrow L(b\omega_1 + a\omega_2)$$

$$(p, u; q, v) \longmapsto (p, q; u^{\otimes b} \otimes v^{\otimes a})$$

This map is compatible with the trivializations and gives rise to a globally defined map  $\phi$ . This can be easily verified using the transition functions in each of the corresponding bundles, for instance:

$$\left. \begin{array}{ll} u_l = e^{2\pi i \theta_{lj}} u_j & \longrightarrow u_l^{\otimes b} = e^{2\pi i b \theta_{lj}} u_j^{\otimes b} \\ v_m = e^{2\pi i \theta_{mk}} v_k & \longrightarrow v_m^{\otimes a} = e^{2\pi i a \theta_{mk}} v_k^{\otimes a} \end{array} \right\} \implies u_l^{\otimes b} \otimes v_m^{\otimes a} = e^{2\pi i a \theta_{lj}} e^{2\pi i a \theta_{mk}} u_j^{\otimes b} \otimes v_k^{\otimes a}$$

Further, the map  $\phi$  is invariant with respect to the action  $\varphi_{(a,-b)}$ . Indeed,

$$\begin{aligned} \phi_{ij}(\varphi_{(a,-b)}(\theta; p, q; u, v)) &= \phi_{ij}(p, q; e^{2\pi i a \theta} u, e^{-2\pi i b \theta} v) = \\ &= (p, q; e^{2\pi i a b \theta} u^{\otimes b} \otimes e^{-2\pi i a b \theta} v^{\otimes a}) = (p, q; u^{\otimes b} \otimes v^{\otimes a}) = \phi_{ij}(p, q; u, v) \end{aligned}$$

This construction restrict to the sphere bundles. It is left to prove that the pre-image of a point through  $\phi$  consist of precisely one orbit. The points belonging to the same orbit verify

$$(u, v) = (e^{2\pi i a \theta} u, e^{2\pi i b \theta} v), \quad \text{for some } \theta \in \mathbb{S}^1$$

Thus there exists  $s, t \in \mathbb{Z}$  such that

$$\theta = \frac{s}{a}, \quad \theta = \frac{t}{b}$$

Since  $(a, b) = 1$  both  $s, t$  are multiples of  $a$  and  $b$  respectively. In consequence, the map  $\phi$  contracts precisely one orbit to a point realizing the manifold  $\mathbb{S}_{(b,a)}(W_1 \times W_2)$  as the orbit space of the action  $\varphi_{(a,-b)}$ . The statement about the contact structure can be explicitly verified by pushing forward the codimension-2 direct sum distribution through the action.  $\square$

*Proof.* (Alternative) Instead of using local coordinates one may translate the statement in terms of principal bundles. Let  $G = \mathbb{S}^1 \times \mathbb{S}^1$  and  $H \cong \mathbb{S}^1 \subset G$  be the subgroup defined as the image of the embedding

$$\varphi_{(a,-b)} : \mathbb{S}^1 \longrightarrow H \subset G, \quad \sigma \longmapsto (a\sigma, -b\sigma).$$

Let  $P$  be the  $G$ -principal bundle with base space  $W_1 \times W_2$  induced by the  $\mathbb{S}^1$ -principal bundles  $\mathbb{S}_1(W_1)$  and  $\mathbb{S}_1(W_2)$ . Our aim is to describe  $P/H$  as a bundle over  $W_1 \times W_2$ . In general  $P \longrightarrow P/H$  is not a  $H$ -principal bundle but it is the case when both  $G$  and  $H$  are closed Lie groups. Actually, they are abelian and since  $(a, b) = 1$ ,  $P/H$  is also a  $G/H$ -principal bundle over  $W_1 \times W_2$ . Taking into account the exact group sequence

$$1 \longrightarrow \mathbb{S}^1 \cong H \longrightarrow G \longrightarrow G/H \cong \mathbb{S}^1 \longrightarrow 1$$

where the second morphism is given by multiplication by  $(b, a)$ , we conclude that the space of orbits  $P/H$  is a manifold diffeomorphic to  $\mathbb{S}_{(b,a)}(W_1 \times W_2)$ . From this point of view the connection and curvature are deduced from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(a,-b)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(b,a)^t} \mathbb{Z} \longrightarrow 0$$

$\square$

**Remark 2.4.** *In both arguments the condition  $(a, b) = 1$  plays an essential role. In the first case the pre-image of a point would contain more than one orbit, whereas in the second the bundle  $P/H$  would not be principal for the action would have non-trivial isotropy groups. Note that the sequence between the free abelian groups is exact if and only if  $(a, b) = 1$ , otherwise there would be a torsion subgroup in the quotient group.*

**Examples:** There are a few simple cases worth mentioning,

- (1) Let  $W_1 = \{pt.\}$  and  $W_2$  arbitrary. Then neither the topology of the resulting space nor the contact structure depend on  $b$ . Indeed,  $\mathbb{S}^1 \times \mathbb{S}_1(W_2)/\sim$  is diffeomorphic to

$$\mathbb{S}_{(b,a)}(pt. \times W_2) \cong \mathbb{S}_a(W_2).$$

Analogously the parameter  $a$  is vacuous if  $W_2 = \{pt.\}$ . In particular  $\mathbb{S}^1 \times \mathbb{S}^1$  quotiented by any  $(a, -b)$  coprime  $\mathbb{S}^1$ -action is diffeomorphic to  $\mathbb{S}^1$ .

- (2) Let  $W_1 = W_2 = \mathbb{CP}^1$  be symplectic manifolds with the Fubini-Study form. Then the space  $\mathbb{S}_{b,a}(\mathbb{CP}^1 \times \mathbb{CP}^1)$  is diffeomorphic to  $\mathbb{S}^3 \times \mathbb{S}^2$  regardless of the values  $a, b \in \mathbb{Z}^+$ , see [WZ] for a proof of this fact. Further, the symplectic structure of the associated line bundle depends only on  $a - b$ . Note that there is an alternative construction of a contact structure in  $\mathbb{S}^3 \times \mathbb{S}^2$  using an open book decomposition with  $T^*\mathbb{S}^2$  pages and an even power of a Dehn twist as monodromy, however such a procedure may only produce vanishing first Chern class and is thus different from  $\mathbb{S}_{b,a}(\mathbb{CP}^1 \times \mathbb{CP}^1)$  if  $a \neq 1$ . See [Ko].

- (3) The previous example can be generalized to construct contact structures on  $\mathbb{S}^{2n+1} \times \mathbb{S}^2$ . Indeed the result implies that the total space of  $\mathbb{S}_{(1,k)}(\mathbb{CP}^n \times \mathbb{CP}^1)$  is a  $\mathbb{S}^{2n+1}$ -bundle over  $\mathbb{S}^2$ . This is because the parameter 1 in the second term of the action allows to produce the fibration. The Hopf action is explicit enough for the classifying map to be described as the element

$$(n+1)k \in \mathbb{Z}_2 \cong \pi_1(SO(2n+2)).$$

Consequently the resulting manifold is diffeomorphic  $\mathbb{S}^{2n+1} \times \mathbb{S}^2$  if  $n$  is odd or  $k$  is even and the complex structure again depends only on  $a - b$ . This result can also be obtained using the standard open book decomposition of  $\mathbb{S}^{2n+1}$  in  $B^{2n}$ -pages and stabilizing it with a  $S^2$ -product.

**Remark 2.5.** *If  $M^n$  is a general contact manifold with  $n \geq 5$ , it is still an open problem to construct a contact structure on  $M \times \mathbb{S}^2$ , preferably homotopic to the natural induced almost contact structure.*

It will be essential for the contact blow-up construction to be able to extend a connection on a submanifold to a global one, let us prove that this is possible under suitable conditions. Recall that a connection  $A$  whose curvature is  $-i\omega$  is not unique: the set of possible choices is parametrized by the infinite dimensional vector space of closed 1-forms. If  $A_0, A_1$  are two connections,  $A_0 - A_1$  is a flat connection on the trivial bundle, that is to say  $A_0 - A_1$  can be viewed as a closed 1-form. So given a fixed connection  $A_0$  on  $L$ , any other connection is of the form  $A_0 + \beta$ , with  $\beta$  a closed 1-form. We can now establish the following:

**Lemma 2.6.** *Let  $S$  be a closed submanifold of  $(M^{2n}, \omega)$ , possibly with smooth boundary, and  $L$  a line bundle on  $M$ . Assume that the restriction morphism  $H^1(M) \rightarrow H^1(S)$  is surjective and let  $A_S$  be a connection over  $L|_S$  whose curvature is  $-i\omega$ . Then, there is a connection  $A$  on  $L$  with curvature  $-i\omega$  such that its restriction to  $S$  is  $A_S$ .*

*Proof.* Let  $A_0$  be a connection on the line bundle  $L \rightarrow M$  with curvature  $-i\omega$ . Denote  $i : S \rightarrow M$ , then  $A_S - i^*A_0 = \beta_S$  is a closed 1-form over  $S$ . In order to complete our argument we need to extend  $\beta_S$  to a global closed 1-form.

By hypothesis the map  $H^1(M) \rightarrow H^1(S)$  is a surjection. Therefore there exists a cohomology class  $[\beta]$  on  $H^1(M)$ , such that restricted to  $S$  coincides with  $[\beta_S]$ . Its difference over  $S$  will be the trivial class on  $H^1(S)$ , so  $\beta_S - i^*\beta = dH_S$ , for some smooth function  $H_S : S \rightarrow \mathbb{R}$ . We extend  $H_S$  to a global smooth function  $H : M \rightarrow \mathbb{R}$ . The form  $A_0 + \beta + dH$  is the required global connection with curvature  $-i\omega$  and extending  $A_S$ .  $\square$

### 3. SURGERY ALONG TRANVERSE LOOPS

In this section we recall the construction from Section 5 in [CPP]. A contact operation, the *contact blow-up*, was defined for tranverse embedded loops. Topologically it consists of a surgery along the loop, the interior of  $\mathbb{S}^1 \times B^{2n}$  is removed and a tubular neighbourhood of the  $(2n-1)$ -sphere  $B^2 \times \mathbb{S}^{2n-1}$  is glued along the common boundary  $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$ . The sphere  $\{0\} \times \mathbb{S}^{2n-1}$  whose neighbourhood is attached is called the exceptional divisor. Let us describe the contact structure obtained in the resulting manifold.

In order to understand the change of the contact structure, consider the manifold

$$T = \mathbb{S}^1 \times (0, 1) \times \mathbb{S}^{2n-1}, \text{ with coordinates } (\theta, r, \sigma).$$

Let  $\alpha_{std}$  be the standard contact form for the standard contact structure on  $\mathbb{S}^{2n-1}$  induced by the complex tangencies. One may consider the following two 1-forms in  $T$ :

$$(1) \quad \eta = d\theta - r^2 \alpha_{std}, \quad \lambda = r^2 d\theta + \alpha_{std}.$$

We have chosen the multiplying functions to be  $r^2$  but any strictly increasing function on  $r$  would do. Both  $\eta, \lambda$  are contact forms, the essential difference lies in their behaviour at *infinity*. To be precise,  $\eta$  is well-behaved with respect to the central loop  $\mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times B^{2n}$  and induces a contact structure on that completion  $\mathbb{S}^1 \times B^{2n}$  whereas  $\lambda$  is a contact form in the completion  $B^2 \times \mathbb{S}^{2n-1}$ . A heuristic argument for this is provided by the induced volume forms:

$$\eta \wedge (d\eta)^n \sim r^{2n-1} \cdot dr \wedge \alpha_{std} \wedge (d\alpha_{std})^{n-1} \wedge d\theta \sim dVol_{B^{2n}} \wedge d\theta$$

$$\lambda \wedge (d\lambda)^n \sim r \cdot dr \wedge d\theta \wedge \alpha_{std} \wedge (d\alpha_{std})^{n-1} \sim dVol_{B^2} \wedge \alpha_{std} \wedge (d\alpha_{std})^{n-1}$$

The multiplying factors  $r^2$  will be referred as the Hamiltonians from now on, since they are the Hamiltonian functions associated to the Hamiltonian parallel transport of the contact fibration  $B^2 \times \mathbb{S}^{2n-1} \rightarrow B^2$ . See [Pr2] for a geometric understanding of the contact structures through their Hamiltonians. Thus, to perform a *contact* surgery we should be able to switch the Hamiltonian from one factor to the other. This is achieved through the Hopf action. Indeed, given  $l \in \mathbb{Z}$  the diffeomorphism

$$(2) \quad \begin{array}{ccc} \phi_l : & T & \longrightarrow T \\ & (\theta, r, z) & \longrightarrow (\theta, r, e^{2\pi i l \theta} z) \end{array}$$

pulls-back the contact form  $\eta$  to  $\bar{\lambda} = (-r^2) \cdot [(l - r^{-2})d\theta + \alpha_{std}]$ .

In order to match  $\ker \lambda$  and  $\ker \bar{\lambda}$  we need the radius to be large enough. This is the step left to prove the following

**Theorem 3.1.** (*Thm. 5.1 in [CPP]*) *Let  $(M^{2n+1}, \xi)$  be a contact manifold. Let  $S \subset M$  be a smooth transverse loop in  $M$ . There exists a manifold  $\bar{M}$  satisfying the following conditions:*

- *There exists a contact structure  $\bar{\xi}$  on  $\bar{M}$ , i.e. the blow-up of  $M$  along  $S$  is a contact manifold.*
- *There exists a codimension-2 contact submanifold  $E$  inside  $\bar{M}$  with trivial normal bundle. Moreover the manifold  $E$  is contactomorphic to the sphere  $\mathbb{S}^{2n-1}$ , with its standard contact structure.*
- *The manifolds  $(M \setminus S, \xi)$  and  $(\bar{M} \setminus E, \bar{\xi})$  are contactomorphic.*

*The manifold  $(\bar{M}, \bar{\xi})$  will be called the contact surgery blow-up of  $M$  along  $S$ . The contact submanifold  $E$  is called the exceptional divisor.*

*Proof.* Consider a tubular neighbourhood of the embedded loop with the contact structure  $\eta$  as in (1), for radius  $r \leq \varepsilon$ . We enlarge this tubular  $\varepsilon$ -neighbourhood using the squeezing technique from [EKP] to obtain a radius 2 neighbourhood. The explicit statement is the following

**Lemma 3.2.** (*Proposition 1.24 in [EKP]*) *For any integer  $k > 0$  and radius  $0 < R_0$ , the following map is a contactomorphism*

$$\begin{aligned} \psi_k : \mathbb{S}^1 \times B^{2n}(R_0) &\longrightarrow \mathbb{S}^1 \times B^{2n}\left(\frac{R_0}{\sqrt{1+kR_0^2}}\right) \\ (\theta, r, w_1, \dots, w_n) &\longrightarrow \left(\theta, \frac{r}{\sqrt{1+kr^2}}, e^{2\pi i k \theta} w_1, \dots, e^{2\pi i k \theta} w_n\right), \end{aligned}$$

and it restricts to the identity at  $\mathbb{S}^1 \times \{0\}$ .

We particularize this lemma at  $R_0 = 2$  and then we need  $k$  large enough to satisfy

$$\frac{2}{\sqrt{1+4k}} < \varepsilon.$$

Therefore we may assume that the tubular neighbourhood for which the standard equation (1) holds for  $\eta$  has radius  $r = 2$ . In the annulus corresponding to radius  $(3/2, 2)$  use  $\phi_1$  to induce the contact structure given by  $\ker \bar{\lambda}$ . Declare  $\ker \lambda$  to define the contact structure in the radius area  $[0, 1/2]$ . It is left to find a strictly increasing Hamiltonian interpolating between  $r^2$  and  $1 - r^{-2}$  in the middle region. This can be done, see Figure 1.

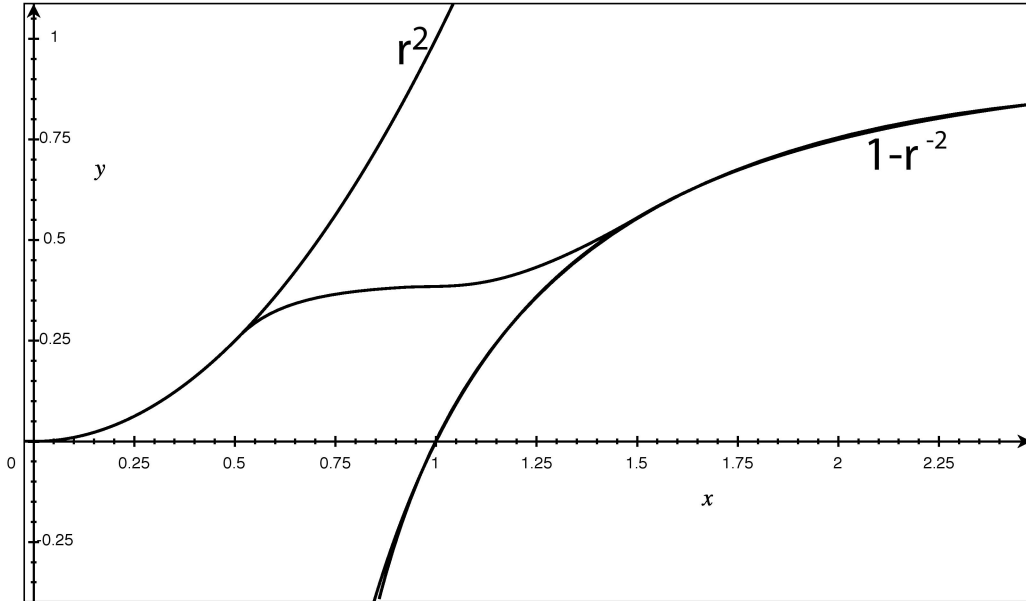


FIGURE 1. Interpolation matching  $\lambda$  and  $\bar{\lambda}$ .

□



**Remark 3.3.** *The process described in the proof can be modified to include the radius squeezing in the gluing map. It suffices to use  $\phi_l$  as gluing map instead of  $\phi_1$  in the domain. Indeed, denote  $T_\rho = \mathbb{S}^1 \times (0, \rho) \times \mathbb{S}^{2n-1}$  and consider the contact structures*

$$\xi_0 = \ker\{d\theta - r^2\alpha_{std}\}, \quad \xi_l = \ker\{(l - r^{-2})d\theta + \alpha_{std}\}.$$

Define the map

$$\varphi : T_2 \mapsto T_{\varepsilon(k)}, \quad (\theta, r, z) \mapsto \left( \theta, \frac{r}{\sqrt{1 + kr^2}}, z \right),$$

where  $\varepsilon(k)$  is the obvious radius in the image. Then the following diagram is commutative in the contact category :

$$\begin{array}{ccc} (T_2, \xi_1) & \xrightarrow{\phi_1} & (T_2, \xi_0) \\ \downarrow \varphi & & \downarrow [\text{EKP}] \\ (T_{\varepsilon(k)}, \xi_l) & \xrightarrow{\phi_l} & (T_{\varepsilon(k)}, \xi_0) \end{array}$$

where Lemma 3.2 is performed with parameter  $k = l - 1$ .

Note that the contactomorphism type of the exceptional divisor is that of the standard sphere, the parameter in the construction allows us to discretely vary the radius of the tubular neighbourhood we are collapsing. From the smooth point of view we have

**Lemma 3.4.** *The maps  $\phi_l$  and  $\phi_k$  are smoothly isotopic if and only if  $k - l$  is even.*

*Proof.* There is a natural morphism

$$\Psi : \pi_1(SO(2n)) \rightarrow \pi_0(\text{Diff}(\mathbb{S}^1 \times \mathbb{S}^{2n-1})),$$

defined as  $\Psi(\gamma_t)(\theta, z) = (\theta, \gamma_\theta(z))$ ,  $t \in \mathbb{S}^1$ . Let  $\gamma_k$  be  $k$ -times the standard circle action on  $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$  induced by  $\mathbb{C}^*$ , then it is clear that  $\phi_k$  is realized as  $\Psi(\gamma_k)$ . Recalling the classical computation of the group  $\pi_1(SO(2n)) \simeq \mathbb{Z}_2$  we conclude that  $\gamma_k = \gamma_l$  if and only if  $k - l$  is even.

It is left to prove that  $\phi_0$  and  $\phi_1$  are not isotopic. In order to do so, we construct two manifolds  $X_0$  and  $X_1$  by gluing two copies of the manifold with  $B^2 \times \mathbb{S}^{2n-1}$  along its boundary by using  $\phi_0$  and  $\phi_1$  respectively. These manifold are not diffeomorphic. Indeed,  $X_0$  has a spin structure since  $w_2(X_0) = 0$ , which is readily seen using the product formula and the fact that any sphere is also a spin manifold. The manifold  $X_1$  is not spin though. This can be seen using the homology class of the section  $s_N$  of the twisted bundle  $X_1 \rightarrow \mathbb{S}^2$  which maps to the north poles: since the tangent bundle of a neighbourhood of its image is not trivial we conclude that  $\langle w_2(X_1), s_N \rangle \neq 0$ . Hence  $\phi_0$  and  $\phi_1$  are not isotopic.  $\square$

Therefore, the smooth type of the blown-up manifold will depend on the parity of the positive integer fixed for the construction. As for the contact type, it is clear that the contact structure are *a priori* non-contactomorphic since by Theorem 1.2 in [EKP] the maps  $\phi_k$  and  $\phi_l$  are not contact compactly supported isotopic for  $k$  different from  $l$ . This does not imply that the contact structure is different. In fact, they could happen to be the same, but at least there is no local contactomorphism relating the two contact structures. It would be nice to find an example of two blow-ups leading to two different contact structures.

## 4. GROMOV'S APPROACH

In this section we develop a more general notion of contact blow-up along a general Boothby–Wang submanifold using the approach suggested by M. Gromov. There is a minimum radius in the size of the tubular neighbourhood of the blown-up submanifold that appears in the construction. We will show in the next section a way to handle this feature by introducing a more functorial definition.

**4.1. Symplectic blow-up.** To begin with we review the definition of the blow-up in the symplectic case as in [MS]. Let  $S \subset (M, \omega)$  be a symplectic submanifold of codimension greater than 2. Denote as  $pr : \nu(S) \rightarrow S$  the symplectic normal bundle of the submanifold  $S$  in  $M$ . The symplectic structure in the bundle is the canonical one associated to the symplectic linear structure in the fibers. We choose a compatible complex structure in the bundle  $\nu(S)$  which, in turn, induces a metric.

The normal bundle  $\nu(S)$  can be considered as an associated vector bundle of a  $U(k)$ –principal bundle  $P \rightarrow S$ . Since the fiber of  $\nu_S$  is symplectomorphic to  $(\mathbb{R}^{2k}, \omega_0)$ , the blow-up operation can be performed in a parametrically linear manner and hence we first focus on the linear case. The symplectic blow-up of the symplectic vector space  $(\mathbb{R}^{2k}, \omega_0)$  at the origin is a symplectic manifold  $\widetilde{\mathbb{R}}_\delta^{2k}$  that satisfies the following properties:

- (1)  $\widetilde{\mathbb{R}}_\delta^{2k} \xrightarrow{\pi} \mathbb{R}^{2k}$  is topologically the complex blow-up at the origin.
- (2) For any small  $\epsilon > 0$ ,  $\widetilde{\mathbb{R}}_\delta^{2k} \setminus \pi^{-1}(B(\delta + \epsilon))$  is symplectomorphic to  $\mathbb{R}^{2k} \setminus B(\delta + \epsilon)$ .
- (3) The exceptional divisor is  $\mathbb{CP}^{k-1}$  with induced symplectic form  $\delta \cdot \omega_{\mathbb{CP}^1}$ .
- (4) The unitary group  $U(k)$  acts Hamiltonially in  $(\widetilde{\mathbb{R}}_\delta^{2k}, \widetilde{\omega}_\delta)$ .

Hence there is an induced bundle  $\widetilde{pr} : \widetilde{\nu}_{S,\delta} \rightarrow S$ . Let  $\beta$  be a connection in  $P$ , there are induced coupling forms  $\alpha$  and  $\widetilde{\alpha}_\delta$ , in  $\nu_S$  and  $\widetilde{\nu}_{S,\delta}$  respectively, restricting to the symplectic form on each fiber and coinciding away from the radius  $\delta + \epsilon$ , see Thm. 6.17 in [MS]. Define the forms

$$\begin{aligned} \omega_\nu &= \alpha + pr^* \omega_S, \\ \widetilde{\omega}_\nu &= \widetilde{\alpha}_\delta + \widetilde{pr}^* \omega_S. \end{aligned}$$

on the bundles  $\nu_S$  and  $\widetilde{\nu}_{S,\delta}$ . They are symplectic forms close to the zero section and to the exceptional divisor respectively. They also coincide away from a neighbourhood of radius  $\delta + \epsilon$ . By the symplectic neighbourhood theorem there is a neighbourhood  $N$  of the symplectic submanifold  $S$  and a symplectomorphism  $\Psi : N \rightarrow U_{\delta_0}$ , where  $U_{\delta_0} := P \times_{U(k)} B(\delta_0)$  is a neighbourhood of the zero section of the symplectic normal bundle. If we now perform a symplectic blow-up with the radius condition  $0 \leq \delta + \epsilon < \delta_0$  on  $\nu_S$  then it can be glued back to the manifold using the symplectomorphism  $\Psi$ .

It is essential to realize that the radius of the tubular neighbourhood of  $S$  cannot be estimated *a priori*, and therefore the symplectic volume of the exceptional divisor cannot be assumed to be arbitrarily large. This will be an obstruction to develop the Gromov's approach in general.

The symplectic blow-up will be of particular interest in the following case: let  $V$  be a symplectic vector bundle over a symplectic manifold  $(W, \omega)$ , then the total space is symplectic as well. Thus, we are able to blow-up the symplectic manifold  $V$  along its zero section  $W$ . Moreover, if the symplectic form  $\omega$  is of integer class, the symplectic form in the resulting blow-up will be of integer class if the radius of the neighbourhood of the zero section that we collapse in  $V$  is a positive integer.

**4.2. Definition of Contact Blow-up.** We are now in a position to define the notion of contact blow-up in terms of the previous description of the symplectic blow-up. This is the definition that in the introduction was referred as *à la Gromov*. Let  $(M, \xi)$  be a contact manifold and  $(S, \xi_S)$  a contact submanifold.

In order to be able to use the symplectic blow-up, we assume that  $S$  is contactomorphic to the total space of a Boothby–Wang fibration  $\pi : \mathbb{S}(W) \rightarrow W$ , for some symplectic manifold  $(W, \omega)$ . To construct the blow-up along a submanifold it is crucial to control its neighbourhoods and the complements (of a small neighbourhood) of the submanifold in them. More precisely, so as to provide a suitable model for the complement we also assume that there exists a symplectic bundle  $V$  over  $W$  such that  $\nu_M(S) \cong \pi^*(V)$ . Observe that the total space of  $V$  carries a symplectic form  $\bar{\omega}$  in the same cohomology class of  $[\omega]$ , under the natural identification of  $H^2(V, \mathbb{R})$  with  $H^2(W, \mathbb{R})$ . As stated previously there exists a symplectic manifold  $(\tilde{V}, \bar{\omega}_W)$  obtained by blowing up  $V$  along its zero section  $W$ ; let  $E$  denote the exceptional divisor of this blow-up. The contact blow-up will be defined under these assumptions.

The following diagram shows the setup described above along with the Boothby–Wang fibrations of the symplectic manifolds  $V$  and  $E$ . The construction of the contact blow-up will be immediate once the relations between the structures involved are clarified:

$$\begin{array}{ccccc}
 \nu_M(S) \cong \pi^*(V) & \mathbb{S}(V) & \mathbb{S}(\tilde{V}) \supset \mathbb{S}(E) = \mathbb{S}(\tilde{V})|_E & & \\
 \downarrow & \downarrow & \downarrow & & \\
 (S, \xi_S) \cong \mathbb{S}(W) & (V, \bar{\omega}) & \longleftarrow (\tilde{V}, \bar{\omega}_W) \supset E & & \\
 \downarrow \pi & \swarrow & & & \\
 (W, \omega) & & & & 
 \end{array}$$

In order for the diagram to even be defined there is a requirement: the symplectic form in  $\tilde{V}$  must be an integral class. This is equivalent to the parameter multiplying the class of the exceptional divisor in the symplectic blow-up being a positive integer. Assume so onwards, we shall later discuss a procedure to ensure this condition in some relevant situations.

Let us first focus on the relation between  $\mathbb{S}(W)$  and  $\mathbb{S}(V)$ . The choice of symplectic form on  $V$  implies that there exists a symplectic embedding of  $W$  in  $V$  and therefore  $\mathbb{S}(W)$  is contained in  $\mathbb{S}(V)$ . This setup also implies that  $\mathbb{S}(W)$  is actually a contact submanifold of  $\mathbb{S}(V)$ . Indeed, the tubular neighbourhood theorem tells us that the normal bundle  $\nu_M(S)$

will be diffeomorphic to a small neighbourhood of  $S$  in  $M$ , but  $\nu_M(S) \cong \pi^*(V)$  so the same situation applies to  $\mathbb{S}(W)$  in  $\mathbb{S}(V)$ .

Analogously, we can study the complement of  $\mathbb{S}(W)$  in  $\mathbb{S}(V)$ . We assert that it is diffeomorphic to the complement of  $S$  in  $\nu_M(S)$ , hence diffeomorphic to a small neighbourhood of the boundary of a tubular neighbourhood of  $S$  in  $M$ . Yet again, this follows from the contact neighbourhood theorem and it applies since the symplectic normal bundle  $\nu_M(S)$  is symplectically isomorphic to  $\nu_{\mathbb{S}(V)}(\mathbb{S}(W))$ .

Therefore, since  $\mathbb{S}(W) \subset \mathbb{S}(V)$  provides a local model, we only need to perform the blow up of  $V$  along  $W$  and study whether the Boothby–Wang structures associated to them allow us to glue back the resulting blow-up model to  $M$ . This is the content of the following:

**Proposition 4.1.** *Let  $S = \mathbb{S}(W)$  be a Boothby–Wang contact submanifold of  $\mathbb{S}(V)$ . Suppose we symplectically blow-up  $W \subset V$  by collapsing a radius 1 neighbourhood. Then, there is a choice of contact form for  $\mathbb{S}(\tilde{V})$  such that  $\mathbb{S}(E)$  is a contact submanifold of  $\mathbb{S}(\tilde{V})$  and the complement of an arbitrary small neighbourhood of  $\mathbb{S}(E)$  in  $\mathbb{S}(\tilde{V})$  is contactomorphic to the complement of some small neighbourhood of  $\mathbb{S}(W)$  in  $\mathbb{S}(V)$ .*

For the sake of a clearer exposition the proof of this proposition will be given at the end of this subsection. Further assume that we are able to choose a tubular neighbourhood  $U'$  of  $\mathbb{S}(W)$  in  $\mathbb{S}(V)$  with radius larger than 1 which is contactomorphic to a tubular neighbourhood  $U$  of  $S$  in  $M$ . Gluing along the boundary of  $U$  a small neighbourhood of  $\mathbb{S}(E)$  in  $\mathbb{S}(\tilde{V})$  produces a manifold  $M'$  with a contact structure which is contactomorphic to  $M$  away from small neighbourhoods of  $\mathbb{S}(E)$  and  $S$  respectively. Finally, we introduce the main definition of this subsection:

**Definition 4.2.** *A contact manifold  $(M', \xi')$  obtained from  $M$  and  $S$  using the procedure above is called a contact blow-up of  $M$  along the contact submanifold  $S$ .*

The exceptional divisor of the contact blow-up is defined to be  $\mathbb{S}(E)$ , where  $E$  is the exceptional divisor of the symplectic blow-up over which is locally modelled. Observe that for the definition to work we need  $S$  to have a tubular neighbourhood of radius at least 1 inside  $M$ .

The most simple example of contact blow-up is the case of a transverse loop  $K$  in  $(M^5, \xi)$ . The loop is contactomorphic to  $\mathbb{S}(pt)$ . Moreover, since the normal bundle is trivial, it is the pull-back of the trivial bundle over the point. Therefore such a case fits in the initial setup of the construction. The symplectic model corresponds to the blow-up of  $\mathbb{C}^2$  at the origin, collapsing a neighbourhood of radius 1, and therefore  $E = \mathbb{CP}^1$ . So, we have that  $\mathbb{S}(E) = \mathbb{S}(\mathbb{CP}^1)$ , i.e. the standard contact 3-sphere. This particular case can be seen, at least topologically, as a surgery along a loop.

If we symplectically blow-up with collapsing radius  $k \in \mathbb{N}^*$ , we obtain that the exceptional divisor is  $\mathbb{S}(\mathbb{CP}^1, k\omega_{\mathbb{CP}^1})$ , i.e. the sphere bundle associated to the polarization  $\mathcal{O}(k)$  of  $\mathbb{CP}^1$ ,

which is the lens space  $L(k, 1)$  with its standard contact structure. Therefore, even the diffeomorphism type of the blown-up contact manifold changes with the blow-up radius  $k \in \mathbb{N}^*$ .

Note that there is not natural projection map from  $\mathbb{S}(E)$ , the exceptional divisor, to the blow-up locus  $\mathbb{S}(W)$ . Already in the case of a loop in a 5-dimensional manifold, the exceptional divisor for a radius 1 blow-up is  $\mathbb{S}^3$  and the blow-up locus is the circle  $\mathbb{S}^1$ . This is a difference with respect to the symplectic and algebraic cases where the exceptional divisor is a bundle over the submanifold along which the blow-up is performed. It is true though that there is a natural projection  $\mathbb{S}(E) \rightarrow E \rightarrow W$ , but it does not lift to  $\mathbb{S}(W)$ .

**Remark 4.3.** *The assumption of the integer radius can be fulfilled in certain cases. For instance in the blow-up along a transverse  $\mathbb{S}^1$  we can use the Lemma 3.2. Therefore the construction in this case will have two natural parameters: the integer radius that determines the topology of the exceptional divisor, and the choice of framing in the spirit of the lemma. The above described construction à la Gromov does not show in general the appearance of this second positive integer, this is a reason to introduce a third way of defining the blow-up highlighting these two choices.*

To conclude this subsection we prove the assertion that allowed us to glue the Boothby–Wang construction over the exceptional divisor in the contact blow-up construction.

*Proof of Proposition 4.1.* We need to find an appropriate connection on the topological Boothby–Wang manifold corresponding to  $\tilde{V}$ .

Notice from the construction of the symplectic blow-up given in [MS] that given a sufficiently small neighbourhood of  $E$  in  $\tilde{V}$  it is possible to choose a symplectic form  $\tilde{\omega}$  on  $\tilde{V}$  such that complement of that neighbourhood in  $\tilde{V}$  is symplectomorphic to a small neighbourhood of  $W$  in  $V$ . Furthermore, observe that the exceptional divisor is just the inverse image of  $W$  contained in  $V$  as the zero section under the blow-up projection  $\phi : \tilde{V} \rightarrow V$ .

Now recall from the previous subsection that the contact structure of  $\mathbb{S}(\tilde{V})$  is determined by the choice of a connection over the associated bundle whose curvature is  $-i\tilde{\omega}$ . So let  $A$  be the connection over  $L$  that determines the contact structure on  $\mathbb{S}(V)$ , and denote by  $U$  an arbitrarily small neighbourhood of  $W$  inside  $V$ . From the construction of the symplectic form  $\tilde{\omega}$  on  $\tilde{V}$  we can assume that the map  $\phi$  is a symplectomorphism of  $V \setminus U$  and  $\phi^{-1}(V \setminus U)$ . Therefore the connection  $\phi^*(A)$  satisfies the required properties over  $\phi^{-1}(V \setminus U)$ . It remains for it to be extended to a connection all over  $\tilde{V}$  with curvature  $-i\tilde{\omega}$ . By Lemma 2.6 such an extension is possible provided that the restriction morphism from  $H^1(\phi^{-1}(V \setminus U), \mathbb{R}) \rightarrow H^1(\tilde{V}, \mathbb{R})$  is surjective. It is sufficient to show that  $\pi_1(\tilde{V}) = \pi_1(\phi^{-1}(V \setminus U))$ .

Indeed, observe that  $\tilde{V}$  is homotopic to a  $(\mathbb{CP}^{r-1})$ -bundle over  $W$ , with  $r \geq 2$ , and hence  $\pi_1(\tilde{V}) = \pi_1(W)$  holds. Note that the manifold  $\phi^{-1}(V \setminus U)$  is diffeomorphic to  $V \setminus U$  and the set  $V \setminus U$  is homotopy equivalent to a sphere bundle over  $W$  with fibers of dimension greater

than 2. From the long exact sequence of homotopy we conclude that

$$\pi_1(\phi^{-1}(V \setminus U)) \cong \pi_1(W).$$

It now follows from Lemma 2.6 that on  $\mathbb{S}(\tilde{V})$  there is a choice of contact form with the required properties. Away from a given small neighbourhood of  $\mathbb{S}(E)$  in  $\mathbb{S}(\tilde{V})$  a contact form can be chosen such that it induces a contactomorphism from the complement of such a neighbourhood to the complement of some neighbourhood of  $W$  in  $V$ , this is because we can choose the symplectic form on  $\tilde{V}$  with the required property.  $\square$

## 5. BLOW-UP AS A QUOTIENT

In this section we define the contact blow-up of a contact manifold  $M$  along a Boothby–Wang contact submanifold  $S$  using the notion of contact cuts. This is a general construction that clarifies the contact structure in the two previous definitions of contact blow-up. We shall first introduce the construction when the normal bundle  $\nu_S$  is trivial and then proceed with the general case.

**Remark 5.1.** *Apart from transverse loops, any isocontact embedding<sup>3</sup> of a contact 3-fold in a sphere will also have trivial normal bundle. This situation does occur, for instance any closed cooriented 3-fold admits an isocontact embedding into the standard contact 7-sphere.*

**5.1. Contact cuts.** We briefly recall the notion of *contact cut* as introduced by E. Lerman in [Le1]. Consider  $\mathbb{S}^1$  as the symmetry Lie group. Given a  $\mathbb{S}^1$ -action on a manifold  $M$ , topologically the cut construction is based on collapsing the boundary of a given submanifold –often described as the level set of a function– with the action.

The quotient will be a smooth manifold if the action is free and the further geometric structure that  $M$  may have is inherited in the cut if reasonable hypotheses hold. Two instances of this are symplectic and contact structures. The case concerning us will be the latter and the result we shall use is the following

**Theorem 5.2.** (*Thm. 2.11 in [Le1]*) *Let  $(M, \alpha)$  be a contact manifold with a  $\mathbb{S}^1$ -action preserving  $\alpha$  and let  $\mu$  denote its moment map. Suppose that  $\mathbb{S}^1$  acts freely on the zero level set  $\mu^{-1}(0)$ . Then the set<sup>4</sup>*

$$M_{[0, \infty)} := \{m \in M \mid \mu(m) \in [0, \infty)\} / \sim$$

*is naturally a contact manifold. Moreover, the natural embedding of the reduced space*

$$M_0 := \mu^{-1}(0) / \mathbb{S}^1$$

*into  $M_{[0, \infty)}$  is contact and the complement  $M_{[0, \infty)} \setminus M_0$  is contactomorphic to the open subset*

$$\{m \in M \mid \mu(m) > 0\} \subset (M, \alpha).$$

Note that the contact reduction requires the regular value to be 0, whereas in the symplectic reduction any regular value is licit. This is so because in the contact reduction it is imposed that the orbits of the isotropy subgroup are tangent to the contact structure, see [Ge2].

<sup>3</sup>The embedding  $e : (M_1, \xi_1) \rightarrow (M_2, \xi_2)$  is isocontact if  $e^*(\xi_2) = \xi_1$ .

<sup>4</sup>The equivalence relation is defined as  $m \sim m' \implies \mu(m) = \mu(m') = 0$  and  $m = \theta \cdot m'$  for some  $\theta \in \mathbb{S}^1$ .

In [Le2] there is a detailed account relating the *symplectic cut* and the *symplectic blow-up* as described in [MS]. It is a suitable point to stress that the notions of symplectic or complex blow-ups are related to *collapsing*, the topology becomes more intricate whereas the *volume* decreases. Although it is not the usual point of view in algebraic geometry, this is a standard fact in topology.

**5.2. Blow-up procedure.** Let  $(S, \ker \alpha)$  be a codimension- $2k$  contact submanifold of  $M^{2n+1}$  with trivial normal bundle. Suppose that  $S \cong \mathbb{S}_a(W)$  for some symplectic manifold  $(W, \omega)$ ,  $a \in \mathbb{Z}^+$ . We will define the *contact blow-up of  $M$  along  $S$* . A tubular neighbourhood of  $S$  is contactomorphic to

$$S_R = S \times B^{2k}(R) \xrightarrow{\text{sph.coord.}} S \times [0, R) \times \mathbb{S}^{2k-1}, \quad \text{for some } R \in \mathbb{R}^+,$$

with the contact structure given by  $\alpha + r^2 \alpha_{std}$ , where  $\alpha_{std}$  is the standard contact form in  $\mathbb{S}^{2k-1}$ . We shall construct the blow-up along  $S$  using the contact cut from the previous subsection. Let  $b \in \mathbb{Z}^+$  and consider the  $\mathbb{S}^1$ -action

$$\begin{aligned} \varphi_{(a,-b)} : \mathbb{S}^1 \times S \times [0, R) \times \mathbb{S}^{2k-1} &\longrightarrow \mathbb{S}(W) \times [0, R) \times \mathbb{S}^{2k-1} \\ (\theta, p, r, z) &\longmapsto ((a\theta) \cdot p, r, e^{-2\pi i b \theta} z) \end{aligned}$$

This action is generated by the field  $X = aR_S - bR_{std}$  where  $R_S, R_{std}$  are the Reeb vector fields associated to  $\alpha$  and  $\alpha_{std}$ . The moment map is

$$\begin{aligned} \mu_{(a,b)} : S \times B^{2k}(R) &\longrightarrow \mathfrak{g}^* \cong \mathbb{R} \\ (p, r, z) &\longmapsto a - br^2 \end{aligned}$$

A smaller neighbourhood of the central fibre  $S$  will be removed and its boundary collapsed as in Figure 2.

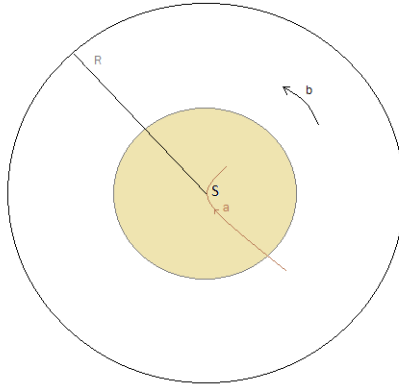


FIGURE 2. Setup before performing the contact blow-up.

Since the contact cut can only be performed in the pre-image of the regular value  $0 \in \mathbb{R}$ , it is thus a necessary condition that  $R^2 \geq a/b$ . Note that this can always be achieved if  $b$  is large enough.

**Definition 5.3.** Let  $S \cong \mathbb{S}_a(W)$  be a contact submanifold of  $(M, \xi)$  with fixed trivial normal bundle  $S \times B^{2k}(R)$ . Let  $b \in \mathbb{Z}^+$  be such that  $R^2 \geq a/b$ . The  $(a, b)$ -contact blow-up  $\widetilde{M}_S$  of  $M$  along  $S$  is defined to be the contact cut of  $M$  for the moment map associated to the circle action  $\varphi_{(a, -b)}$  :

$$\widetilde{M}_S := M_{\{\mu_{(a, b)} \leq 0\}}$$

The collapsed region  $\mu_{(a, b)}^{-1}(0)/\sim$  will be called the *exceptional divisor*, it is a contact manifold of dimension  $2n - 1$ . The induced  $\mathbb{S}^1$ -action in the level set

$$\mu_{(a, b)}^{-1}(0) \cong S \times \{\sqrt{a/b}\} \times \mathbb{S}^{2k-1}$$

coincides with the action  $\varphi_{(a, -b)}$  defined in Theorem 2.3 with  $W_1 = W$  and  $W_2 = \mathbb{CP}^{k-1}$ . Thus, the orbit space is

$$\mu_{(a, b)}^{-1}(0)/\mathbb{S}^1 \cong \mathbb{S}_{(b, a)}(W \times \mathbb{CP}^{k-1}) \cong \mathbb{S}(W) \times \mathbb{S}(\mathbb{CP}^{k-1})/\sim.$$

Notice that both the topology and the contact structure of the exceptional divisor strongly depend on the choice of the parameters  $a$  and  $b$ . Consequently, so does  $\widetilde{M}_S$ . Recall the discussion at the end of Section 3.

In the case of a contact 5-fold, a transverse circle –the simplest contact submanifold– is replaced by a (quotient of a) standard contact 3-sphere, as in Section 3. This new construction of the blow-up along a transverse loop will be compared with the previous ones the next section. The next simplest case is to blow-up a contact 3-sphere

$$\mathbb{S}^3 \cong \mathbb{S}(\mathbb{CP}^1) \subset M^{2n+1}.$$

The topology of the exceptional divisor will depend on the element of the corresponding higher homotopy group providing the classifying map as explained in Section 2. If  $M$  is 5-dimensional  $\mathbb{S}^3$  is the exceptional divisor of the  $(1, k)$  blow-up whereas the exceptional divisor of a  $(1, k)$  blow-up along  $\mathbb{S}^3$  is diffeomorphic to  $\mathbb{S}^2 \times \mathbb{S}^{2n-3}$  for  $n \geq 3$  and  $k$  even, cf. Example (3) in Subsection 2. Observe that in general the blow-up along a contact divisor has no effect if  $b = 1$  but does indeed modify the manifold otherwise.

**5.3. Blow-up general normal bundle.** In this section we define the contact blow-up along a contact submanifold with a not necessarily trivial normal bundle. The construction will clearly coincide with the previous blow-up in the case of a trivial normal bundle. We first recall the tubular neighbourhood theorem and list a few instances of contact submanifolds with non-trivial normal bundle.

**5.3.1. Preliminaries.** In smooth topology the smooth structure of a neighbourhood of a submanifold is retained by the normal bundle. Similarly, the contact geometry nearby a contact submanifold  $(S, \xi_S)$  is determined by the normal bundle  $\nu_S$  along with *contact* information. In this case this information is a conformally symplectic structure. Such a structure exists because  $\nu_S$  can be identified with the symplectic orthogonal  $\xi_S^\perp$ .

More precisely, the statement of the contact neighbourhood theorem follows:



**Theorem 5.4.** (2.5.15 in [Ge]) *Let  $(S_1, M_1)$  and  $(S_2, M_2)$  be contact pairs such that  $(S_1, \xi_{S_1})$  is contactomorphic to  $(S_2, \xi_{S_1})$ . If  $\xi_{S_1}^\perp \cong \xi_{S_2}^\perp$  as conformally symplectic bundles, then there exists a contactomorphism between suitable neighbourhoods of  $S_1$  and  $S_2$ .*

**5.4. Examples.** Let us mention a few cases of contact pairs  $(S, M)$  with  $TM/TS$  non-trivial. Topologically, any bundle  $E$  over a compact manifold  $S$  is a normal bundle of an embedding inside a containing compact manifold  $M$ : let  $M = \mathbb{S}(E \oplus \mathbb{R})$  then  $E$  is the normal bundle of the infinity section, which is diffeomorphic to  $S$ . This example extends to the contact setting naturally. Indeed, let  $(M, \xi = \ker \alpha)$  be a cooriented contact manifold and  $\xi$  itself be non-trivial as an abstract vector bundle. Then, the contact form provides a contact embedding  $\alpha : M \rightarrow \mathbb{S}(T^*M)$  such that the normal bundle of the contact submanifold  $M$  is isomorphic to  $\xi$ . Another class of simple examples are the pair of Boothby–Wang manifolds  $(\mathbb{S}(W), \mathbb{S}(V))$  arising from a pair of symplectic manifolds  $(W, V)$ : if  $\nu_V(W)$  is a non-trivial normal bundle, as for the pair  $(W, V) = (\mathbb{CP}^1, \mathbb{CP}^3)$ , neither will be the induced normal bundle of the contact submanifold  $\mathbb{S}(W)$  in  $\mathbb{S}(V)$ .

More involved examples of such contact submanifolds  $S \subset M$  are homologically non-trivial contact submanifolds defined as the vanishing set of a section in  $H^0(M, E)$  since then

$$c_1(\nu(S)) = PD([S]) \neq 0.$$

This occurs for the possible contact Lefschetz pencil decompositions of  $(M, \xi)$ . As explained in [Pr1] there exists a contact Lefschetz pencil such that the fibres are contact submanifolds, away from the critical points, and so is the base locus. In general neither of them do have trivial normal bundle. The contact pencil approach has been used in dimension 5 to prove existence results of contact structures, there only the simplest version of the blow-up along a transverse loop was required. If the proof adapts to higher dimensions, such as 7, the base locus of the pencil is no longer a circle and generically becomes a contact 3-fold with non-trivial normal bundle. Thus the general definition of the blow-up can be used to produce a contact fibration from the contact pencil.

Finally, it might be worth reminding that Gromov’s  $h$ -principle applies to codimension-4 isocontact embeddings. Thus, to produce examples of contact submanifolds with non-trivial normal bundle we may just consider an isocontact embedding

$$(M^{2n+1}, \xi) \rightarrow (\mathbb{S}^{4n+3}, \xi_{std})$$

where  $(M, \xi)$  is a closed cooriented contact manifold, such an embedding does exist after [Gr]. Since the tangent bundle of the spheres are stable, it is simple to give sufficient conditions for the normal bundle to be non-trivial, for instance  $M$  not being orientable or spin.

Once several examples have been provided, let us describe the ingredients for the general model of the blow-up of a contact manifold  $(M, \xi)$  along a Boothby–Wang submanifold  $(S, \xi_S)$ .

**5.5. Definition.** In the blow-up construction for the trivial normal bundle case there are a couple of circle actions. The first one exists on the contact submanifold  $S$ , since we assumed

that it was of Boothby–Wang type, and it is extended to a local neighbourhood coordinate-wise. The second circle action is the gauge action provided by the complex structure in the conformally symplectic normal bundle. The latter is still available in the non-trivial normal bundle case, the former can *a priori* no longer be extended in a neighbourhood.

We hence require a lifting condition for the circle action on  $S$ . The appropriate set-up is depicted as in the diagram found in the section describing Gromov’s approach:

$$\begin{array}{ccc}
 \nu_M(S) \cong \pi^*(V) & & \mathbb{S}(V) \\
 \downarrow & & \downarrow \\
 (S, \xi_S) \cong \mathbb{S}_a(W) & & (V, \overline{\omega}) \\
 \downarrow \pi & \swarrow & \\
 (W, \omega) & & 
 \end{array}$$

where  $V$  is a symplectic bundle over a symplectic manifold  $W$ . Assume  $a = 1$  for simplicity. The circle action can be naturally extended to a neighbourhood of  $S$ . Indeed, being  $W$  a symplectically embedded submanifold of  $V$ ,  $\mathbb{S}(W)$  is a contact submanifold of  $\mathbb{S}(V)$ . The tubular neighbourhood theorem tells us that the normal bundle  $\nu_M(S)$  is diffeomorphic to a small neighbourhood of  $S$  in  $M$ , but after the smooth isomorphism  $\nu_M(S) \cong \pi^*(V)$  the same situation applies to  $\mathbb{S}(W)$  in  $\mathbb{S}(V)$ . Since the isomorphism holds at the level of symplectic bundles, the contact tubular neighbourhood theorem ensures that there exists a contactomorphism  $\Phi$  between a contact neighbourhood of the zero section in  $\nu(S)$  and a contact neighbourhood of  $\mathbb{S}(W)$  in  $\mathbb{S}(V)$ . Consequently, the circle action in  $\mathbb{S}(V)$  can be carried along  $\Phi$  to a neighbourhood of  $S$ .

In order to define the contact blow-up it is sufficient to spell out the moment map of the circle action. Let us call the circle action on the normal bundle induced by its complex structure the *gauge action*. This action is the natural  $\mathbb{S}^1$ -action when working with a contact pair  $(S, M)$ . Further, the radius coordinate  $r \in \mathbb{R}^{\geq 0}$  is a global coordinate regardless of the non-triviality of the normal bundle: the moment map of this action is precisely  $-r^2$ . The remaining action described above will be referred as the *Boothby–Wang action*. It is the natural action when identifying a neighbourhood of  $S$  in  $M$  with a neighbourhood of  $\mathbb{S}(W)$  in  $\mathbb{S}(V)$ . Since this circle action realizes the Reeb vector field in this setting its moment map is 1. As for the trivial normal bundle case the concatenation  $\varphi_{(1,-1)}$  of the two circle actions is the one providing the appropriate contact cut: expressing the  $r$  coordinate through the contactomorphism  $\Phi$  we conclude that the moment map of the resulting  $\mathbb{S}^1$ -action is  $1 - \Phi(r)^2$ .

The contact cut can be performed if 0 lies in the image of the moment map of the action. The same argument using a multiple of the gauge action concludes that we may modify the action in order to ensure this:  $\Phi$  maps the zero section to  $\mathbb{S}(W)$  and thus the values of  $1 - b^2\Phi(r)^2$  form a decreasing sequence in  $b$  that eventually crosses zero,  $\mathbb{R}^{\leq 0}$  being bounded below. The Boothby–Wang action may as well be arranged to period  $a$ : the concatenation action is denoted  $\varphi_{(a,-b)}$ . We are in position to write the

**Definition 5.5.** (*Contact Blow-Up*) Let  $S \cong \mathbb{S}_a(W)$  be a contact submanifold of  $(M, \xi)$ . Let  $a, b \in \mathbb{Z}^+$  be such that the origin is contained in the image of the moment map  $\mu_{(a,b)}$  for the action  $\varphi_{(a,-b)}$ . The  $(a,b)$ -contact blow-up  $\widetilde{M}_S$  of  $M$  along  $S$  is defined to be the contact cut of  $M$  for the action  $\varphi_{(a,-b)}$ , i.e.  $\widetilde{M}_S := M_{\{\mu_{(a,b)} \leq 0\}}$ .

## 6. UNIQUENESS FOR TRANSVERSE LOOPS

In this section we relate the three constructions of the contact blow-up. The construction that can be performed in the most general situation is the one involving the contact cut. It has two degrees of freedom: a pair of positive integers  $a$  and  $b$ . These two parameters relate to previous integers appearing in the first two constructions. Indeed, the parameter  $l$  in the contact surgery blow-up corresponds to  $b$ . For Gromov's construction, the choice of collapsing radius  $k \in \mathbb{Z}^+$  gives rise, in the case of transverse loops, to the exceptional divisor  $L(k, 1)$  and it corresponds to the parameter  $a$ . It is quite obvious that the diffeomorphism type of the blown-up manifolds is the same regardless of the chosen construction as soon as the parameters coincide as just mentioned.

Let us turn our attention to the contact structure: we restrict ourselves to the case of transverse loops. Denote by  $\overline{M}_b$  the surgery contact blow-up defined in Section 3 with parameter  $b$ . The contact blow-up as defined in Section 4 with radius  $a$  is denoted by  $M'_a$ . And  $\widetilde{M}_{(a,b)}$  will be the contact-cut blow-up as defined in Section 5, performed with parameters  $(a, b)$ . Let us show that uniqueness holds in this case, more precisely we prove the following

**Theorem 6.1.** *Let  $(M, \xi)$  be a contact manifold. Performing the blow-up along a fixed transverse loop with the three procedures introduced previously, the resulting blown-up manifolds  $\overline{M}_1$ ,  $M'_1$  and  $\widetilde{M}_{(1,1)}$  endowed with the blown-up contact structures are contactomorphic. Further, given any pair of integers  $(a, b)$ , the following contactomorphisms hold:*

$$(\overline{M}_b, \bar{\xi}_b) \cong (\widetilde{M}_{(1,b)}, \widetilde{\xi}_{(1,b)}), \quad (M'_a, \xi'_a) \cong (\widetilde{M}_{(a,1)}, \widetilde{\xi}_{(a,1)}).$$

The relation between the different constructions is already hinted in Section 3. Since the exceptional contact divisors coincide and the procedure is of a local nature, i.e. the contact manifold is not altered away from a neighbourhood of the embedded transverse loop, the study should focus on the natural annulus contact fibration. Let us review a few facts. A contact fibration is a fibration  $(M, \xi) \rightarrow B$  such that the fibers are contact submanifolds. We consider contact fibrations over the disk  $f : (V, \xi) \rightarrow B^2$ . The base being contractible, the fibration is trivial and we also assume it to be trivialized. Let us introduce the following

**Definition 6.2.** *Let  $(r, \theta)$  be polar coordinates on the disk  $B^2$ . A trivialized contact fibration over the disk  $\pi : F \times B^2 \rightarrow B^2$  is said to be radial if the contact structure admits the following equation*

$$(3) \quad \ker \alpha_0 = \ker \{\alpha_F + Hd\theta\},$$

where  $H : F \times B^2 \rightarrow \mathbb{R}$  is a smooth function such that  $H = O(r^2)$ .

Notice that for the total space of a radial contact fibration to have an induced contact structure it is required that

$$(4) \quad \frac{\partial H}{\partial r} > 0, \text{ for } r > 0.$$

It is convenient to extend the previous definition in order to include the general situation, where lens spaces may appear as exceptional contact divisors:

**Definition 6.3.** *A trivialized radial contact fibration  $\pi : \mathbb{S}^{2n-1} \times B^2 \rightarrow B^2$  is  $\mathbb{Z}_a$ -equivariant if the natural diagonal  $\mathbb{Z}_a$ -action on the fibration preserves the radial contact structure.*

The action in the fiber sphere  $\mathbb{S}^{2n-1}$  is generated by an  $\frac{2\pi}{a}$ -rotation along the Hopf fiber, whereas the action in the base  $B^2$  is the standard  $\frac{2\pi}{a}$ -rotation in the disk. They preserve respectively the standard contact structure in  $\mathbb{S}^{2n-1}$  and the 1-form  $d\theta$  in the disk. Hence, the fibration becomes equivariant if the function  $H$  is preserved by the action.

Topologically it is fairly straightforward that the blow-up operations we are performing are tantamount to a priori different fillings of the fibration over an annulus to form a manifold lying over the disk – this being always considered up to a finite action  $\mathbb{Z}_a$ , for lens spaces fillings. The transition from  $\mathbb{S}^1 \times B^{2n}$  to  $B^2 \times \mathbb{S}^{2n-1}$  can be understood in these terms: both fibrations over the annulus –produced by restricting to  $r \in (0.5, 1)$ – are filled in the origin with a circle and  $\mathbb{S}^{2n-1}$  respectively. The topological property that allows us to do so is the Hopf fibration, that is to say,  $\mathbb{S}^{2n-1}$  being the total space of a circle bundle. In order to understand the behaviour of the contact structures with respect to these fibrations, it is useful to also adopt the Hamiltonian point of view as in Section 3. Intuitively, the Hamiltonians should be interpreted as monodromies, see [Pr2] for an explicit description. In the transverse loop case it will be enough to use the following

**Lemma 6.4.** *Let  $V$  be a manifold with contact structures  $\xi_0$  and  $\xi_1$ . Assume that there are two smoothly isotopic diffeomorphisms*

$$f_0 : V \rightarrow F \times B^2 \text{ and } f_1 : V \rightarrow F \times B^2,$$

*which are contactomorphisms<sup>5</sup> for  $\xi_0$  and  $\xi_1$  respectively. Let the two fibrations be radial contact fibrations with common contact fiber  $F$  and satisfying that the diffeomorphism*

$$f_1 \circ f_0^{-1} : F \times B^2 \rightarrow F \times B^2$$

*is the identity close to the boundary. Then, the contact structures  $\xi_0$  and  $\xi_1$  are isotopic.*

*Further, if the fiber is  $F \cong \mathbb{S}^{2n-1}$  and the contact fibrations are  $\mathbb{Z}_a$ -equivariant, the contact structures are isotopic through  $\mathbb{Z}_a$ -equivariant contactomorphisms.*

*Proof.* This can be reduced to the setup with a fibration  $F \times B^2$  with two different radial contact structures

$$\begin{aligned} \alpha_0 &= \alpha_F + H_0 d\theta, \\ \alpha_1 &= \alpha_F + H_1 d\theta, \end{aligned}$$

---

<sup>5</sup>A priori, not necessarily contact isotopic.

such that the Hamiltonians  $H_0$  and  $H_1$  coincide close to the boundary. In that setting, we just need to construct a path of functions  $H_t : F \times B^2 \rightarrow \mathbb{R}$  connecting them, relative to the boundary, satisfying the contact equation (4) and the condition  $H_t = O(r^2)$ . But this is possible since the space of such functions is convex.

The argument still works in the equivariant case: the only sentence to be added is that the space of equivariant Hamiltonians is also convex.  $\square$

Thus, to conclude uniqueness we study the contact topology of the different blow-up constructions and ensure that the lemma applies. Regarding the smooth topology, the approach in Section 4 can also be described in terms of fibrations over an annulus. Indeed, the initial  $\mathbb{S}^1$ -core of the annulus disappears when working over the symplectic manifold inducing the contact setup through the Boothby–Wang action. It is essentially this same circle action that reappears when we glue the Boothby–Wang neighbourhoods near the exceptional divisor. However, as it occurs in the surgery case, the circle in the core of the annulus over which the resulting manifold fibers is no longer that  $\mathbb{S}^1$  but the boundary of the disk sub-bundle in the symplectic normal bundle of the exceptional divisor. The third procedure, obtaining the blown-up manifold through a contact reduction performs both of these steps at once. At a contact topology level, this exchange in the representatives of the core of the annulus translates into the Hamiltonians multiplying one or the other summands in the contact form. Let us conclude by detailing the uniqueness argument.

*Proof of Theorem 6.1:* Let us describe the common model fibration that underlies the three constructions in this case. Consider a standard contact neighbourhood  $\mathbb{S}^1 \times (0, 2) \times \mathbb{S}^{2n-1}$  of the given fixed loop and the morphism

$$\begin{aligned} \phi_{(a,b)} : (\mathbb{S}^1 \times (0, 2) \times \mathbb{S}^{2n-1}) &\longrightarrow \mathbb{S}^1 \times (0, 2) \times \mathbb{S}^{2n-1} \\ (\theta, r, z) &\longrightarrow (a\theta, r, e^{2\pi i b \theta} z). \end{aligned}$$

It does generalize the diffeomorphism provided by equation (2) that reflects the case  $a = 1$ . If  $a$  is greater than 1, it becomes a  $a : 1$  covering. The covering transformation is provided by  $\mathbb{Z}_a$  acting through:

$$\begin{aligned} \mathbb{Z}_a \times (\mathbb{S}^1 \times (0, 2) \times \mathbb{S}^{2n-1}) &\longrightarrow (\mathbb{S}^1 \times (0, 2) \times \mathbb{S}^{2n-1}) \\ (l, (\theta, r, p)) &\longrightarrow \left( \frac{2\pi l}{a} + \theta, r, e^{2\pi i b l / a} p \right), \end{aligned}$$

which is free as long as  $(a, b) = 1$ . To understand the change in the contact structure, note that the pull-back of the standard contact form  $\eta = d\theta - r^2 \alpha_{std}$  is given by

$$\lambda = \phi_{(a,b)}^* \eta = (-r^2) \cdot [(b - ar^{-2})d\theta + \alpha_{std}].$$

Denote by  $R_0 = (\frac{b}{a})^{1/2}$  the critical radius where the distribution becomes horizontal. In these coordinates, for any fixed small  $\varepsilon > 0$ , the projection onto the first two factors

$$\pi : \mathbb{S}^1 \times (R_0 + \varepsilon, 2) \times \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^1 \times (R_0 + \varepsilon, 2)$$

provides a radial contact fibration on the annulus and since the function  $(ar^{-2} - b)$  is strictly positive in  $(R_0 + \varepsilon, 2)$ , it can be extended to the interior of the disk to a  $\mathbb{Z}_a$ -equivariant radial

contact fibration. In order to glue back the model to the manifold we should quotient the equivariant contact fibration by  $\mathbb{Z}_a$ , this allows us to use the map  $\phi_{(a,b)}$  to insert the model back into the manifold.

It is thus left to verify that the three blow-up procedures provide examples of such an extension for particular values of  $(a, b)$ . Then Lemma 6.4 will apply to provide the uniqueness of the constructions. Note that the contact surgery blow-up construction is by definition a radial contact fibration, with  $a = 1$ , as shown in Section 3. Let us study the two remaining cases.

To understand the proof in Section 4, let us proceed backwards and instead of applying the Boothby–Wang construction, we produce a contact structure and then quotient the resulting contact manifold by the Reeb  $\mathbb{S}^1$ -action to study whether it is the correct object. Once the coordinate change  $\phi_{(a,b)}$  is performed, the Reeb vector field  $\partial_\theta$  becomes

$$\phi_{(a,b)}^*(\partial_\theta) = \frac{1}{a}(\partial_\theta - bR_{std}).$$

This vector field extends to the interior of the disk fibration and so we may quotient the resulting manifold  $B^2 \times \mathbb{S}^{2n-1}$ . We obtain the blown-up symplectic ball  $\tilde{B}^{2n}$  as its quotient. We can further quotient by the free  $\mathbb{Z}_a$ -action to obtain a non-trivial fibration over the disk  $B^2$ . This proves that a suitable choice of connection leads to an equivariant contact fibration.

There are other choices of connection though. From the principal bundle point of view, a radial contact fibration over the annulus  $\mathbb{S}^1 \times (0, 2)$  corresponds to a connection on

$$B^2 \times \mathbb{S}^{2n-1} \longrightarrow \tilde{B}^{2n}.$$

Certainly, after Proposition 4.1 the contact structure is fixed with the choice of a connection. Note that the space of connections is affine and thus, after Gray’s stability theorem, the resulting contact structures are contact isotopic for different choices of connections. In conclusion, this second model also provides an extension of the model fibration.

We describe the third procedure also beginning with the resulting contact manifold and giving the pull-back of the action. This contact cut construction is also an equivariant radial contact fibration since the pull-back of the vector field generating the  $\mathbb{S}^1$ -action, that is

$$X = a\partial_\theta - bR_{std},$$

is expressed as  $\phi_{(a,b)}^*X = \partial_\theta$  after the coordinate change. There, one may easily verify that the contact cut is just an equivariant radial contact fibration.  $\square$

**Remark 6.5.** *Using Lemma 6.4, we can show that the contact blow-up is unique up to the choice of a trivializing chart of the neighbourhood of the transverse loop. In order to prove the uniqueness of the blow-up along transverse loops, we would need to study the space of isocontact embeddings of the contact manifold  $\mathbb{S}^1 \times B^{2n}$  in  $M$ . It is probably false that it is connected, which is the requirement needed to ensure the uniqueness of the blow-up once the parameters  $a, b$  are fixed.*

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